Exp-Function Method for Finding Exact Solutions of Nonlinear Evolution Equations

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DECLARATION

I hereby declare that the thesis entitled "**Exp-Function Method for Finding Exact Solutions of Nonlinear Evolution Equations**" which is being submitted by me to National Institute of Technology for the award of degree of Master of Science is original and authentic work conducted by me in the Department of Mathematics, National Institute of Technology Rourkela, under the supervision of Prof. Santanu Saha Ray, Department of Mathematics, National Institute of Technology, Rourkela. No part or full form of this thesis work has been submitted elsewhere for a similar or any other degree.

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CERTIFICATE

This is to certify that the project report entitled "**Exp-Function Method for Finding Exact Solutions of Nonlinear Evolution Equations**" submitted by **Lakesh Kumar Ravi** for the partial fulfillment of M.Sc. degree in Mathematics, National Institute of Technology Rourkela, Odisha, is a bona fide record of review work carried out by him under my supervision and guidance. The content of this report, in full or in parts, has not been submitted to any other institute or university for the award of any degree or diploma.

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ABSTRACT

In this report, we applied Exp-function method to some nonlinear evolution equations to obtain its exact solution. The solution procedure of this method, by the help of symbolic computation of mathematical software, is of utter simplicity. The prominent merit of this method is to facilitate the process of solving systems of partial differential equations. These methods are straightforward and concise by themselves; moreover their applications are promising to obtain exact solutions of various partial differential equations. The obtained results show that Exp-function method is very powerful and convenient mathematical tool for nonlinear evolution equations in science and engineering.

CONTENTS

INTRODUCTION

Recently a variety of powerful methods has been proposed and analyzed to obtain the exact solutions to nonlinear evolution equations such as the homogeneous balance principle, F-expansion method, $tanh$ method, auxiliary equation method, the coupled Riccati equations, the Jacobi elliptic equations, etc. Among these methods, the Exp-function method introduced by He and Wu in 2006, is successfully applied to many kind of nonlinear problems. Up to now, the Exp-function method has been applied to find the solutions of a class of nonlinear evolution equations, such as the nonlinear Schrodinger equation, KdV equation with variable coefficient, the discrete $(2+1)-$ dimensional Toda lattice equation and the Maccari's system. So it is easy to see that the Exp-function method is very powerful technique and can be used to study the exact solutions of the high-dimensional system, the discrete system and the system with variable coefficients. All applications verified that Exp-function method is straightforward, concise and effective in obtaining generalized solitary solutions and periodic solutions of nonlinear evolution equations. The main merits of this method over other method are that it gives more general solutions with some free parameters. Also this method can be applied to a wide class of nonlinear evolution equations including those in which the odd and even order derivative terms coexist.

CHAPTER 1

1.1 *Basic idea of Exp-Function Method:*

In order to illustrate the basic idea of the Exp-function method, consider a given nonlinear dispersive equation of the form:

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{1.1.1}$$

This equation is called modified KdV equation, which arises in the process of understanding the role of nonlinear dispersion and in the formation of structure like liquid drops, and it exhibits compactors: solitons with compact support.

Introducing a complex variation η defined as

$$\eta = kx + \omega t \tag{1.1.2}$$

We have

$$\omega u' + k u^2 u' + k^3 u''' = 0 \tag{1.1.3}$$

where prime denotes the differential with respect to η.

The Exp-function method is very simple and straightforward, it is based on the assumption that the travelling wave solution can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n e^{n\eta}}{\sum_{m=-p}^{q} b_m e^{m\eta}} \tag{1.1.4}$$

Where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

We suppose that the solution of eq. (1.1.3) can be expressed as

$$u(\eta)=\frac{a_c\exp(c\eta)+\cdots+a_{-d}\exp(-d\eta)}{b_p\exp(p\eta)+\cdots+b_{-q}\exp(-q\eta)} \tag{1.1.5}$$

To determine values of c and p, we balance the linear term of highest order in eq. (1.1.3) with the highest order nonlinear term. By simple calculation, we have

$$u'''=\frac{c_1\exp[(7p+c)\eta]+\cdots}{c_2\exp[8p\eta]+\cdots} \tag{1.1.6}$$

and

$$u^2u'=\frac{c_3\exp[(p+3c)\eta]+\cdots}{c_4\exp[4p\eta]+\cdots}=\frac{c_3\exp[(5p+3c)\eta]+\cdots}{c_4\exp[8p\eta]+\cdots} \tag{1.1.7}$$

where c_i are known coefficients.

Balancing highest order of Exp-function in eqs. (1.1.6) and (1.1.7), we have

$$7p+c=5p+3c$$

which yields $$p=c \tag{1.1.8}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in eq. (1.1.3) with the lowest order nonlinear term.

$$u'''=\frac{\cdots+d_1\exp[-(7q+d)\eta]}{\cdots+d_2\exp[-8q\eta]} \tag{1.1.9}$$

and

$$u^2u' = \frac{\cdots + d_3 \exp[-(q+3d)\eta]}{\cdots + d_4 \exp[-4q\eta]} = \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[-8q\eta]} \tag{1.1.10}$$

where d_i are known coefficients.

Balancing lowest order of Exp-function in eqs. (1.1.9) and (1.1.10), we have

$$-(7q+d) = -(5q+3d)$$

which yield

$$q = d \tag{1.1.11}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so eq. (1.1.5) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \tag{1.1.12}$$

Substituting eq. (1.1.12) into eq. (1.1.3), and by the help of Mathematical software, we have

$$\frac{1}{A}(C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1}\exp(-\eta) + C_{-2}\exp(-2\eta) +$$

$$C_{-3}\exp(-3\eta)) = 0 \tag{1.1.13}$$

where,

$$A = (\exp(\eta) + b_{-1}\exp(-\eta) + b_0)^4$$

$$C_3 = \omega a_1 b_0 + k a_1^3 b_0 - k^3 a_0 - \omega a_0 - k a_1^2 a_0 + k^3 a_1 b_0$$

$$C_2 = 8k^3 a_1 b_{-1} + 2k a_1^3 b_{-1} - 4k^3 a_1 b_0^2 - 2\omega a_{-1} - 2k a_1 a_0^2 + 2\omega a_1 b_{-1} + 4k^3 a_0 b_0 - 2k a_1^2 a_{-1}$$

$$+ 2k a_1^2 a_0 b_0 + 2\omega a_1 b_0^2 - 2\omega a_0 b_0 - 8k^3 a_{-1},$$

$$C_1 = \omega a_1 b_0^3 + 6\omega a_1 b_0 b_{-1} - \omega a_0 b_0^2 - k^3 a_0 b_0^2 - 18k^3 a_1 b_0 b_{-1} - 6k a_1 a_0 a_{-1} + k a_1 a_0^2 b_0 - k a_0^3$$

$$+23k^3 a_0 b_{-1} - \omega a_0 b_{-1} - 5\omega a_{-1} b_0 + k^3 a_1 b_0^3 - 5k^3 a_{-1} b_0 + k a_1^2 a_{-1} b_0 + 5k a_1^2 a_0 b_{-1},$$

$$C_0 = 4\omega a_1 b_{-1}^2 - 4k a_1 a_{-1}^2 + 32k^3 a_{-1} b_{-1} + 4k a_1 a_0^2 b_{-1} - 32k^3 a_1 b_{-1}^2 + k^3 a_1 b_0^2 b_{-1} - 4\omega a_{-1} b_{-1}$$

$$-4k^3 a_{-1} b_0^2 - 4k a_0^2 a_{-1} - 4\omega a_{-1} b_0^2 + 4k a_1^2 a_{-1} b_{-1} + 4\omega a_1 b_0^2 b_{-1},$$

$$C_{-1} = 18k^3 a_{-1} b_0 b_{-1} - 6\omega a_{-1} b_0 b_{-1} - k^3 a_{-1} b_0^3 + k^3 a_0 b_{-1} b_0^2 + \omega a_0 b_{-1}^2 - 5k a_0 a_{-1}^2 + 5\omega a_1 b_0 b_{-1}^2$$

$$+\omega a_0 b_{-1} b_0^2 - \omega a_{-1} b_0^3 - k a_1 a_{-1}^2 b_0 - 23k^3 a_0 b_{-1}^2 - k a_0^2 a_{-1} b_0 + 5k^3 a_1 b_0 b_{-1}^2 + k a_0^3 b_{-1}$$

$$+6k a_1 a_0 a_{-1} b_{-1},$$

$$C_{-2} = 2\omega a_0 b_{-}^2 b_0 - 2\omega a_{-1} b_{-1}^2 - 2k^3 a_{-1} + 2k a_1 a_{-1}^2 b_{-1} + 2\omega a_1 b_{-1}^3 - 4k^3 a_0 b_{-1}^2 b_0 - 2\omega a_{-1} b_0^2 b_{-1}$$

$$+4k^3 a_{-1} b_0^2 b_{-1} - 8k^3 a_{-1} b_{-1}^2 + 2k a_0^2 a_{-1} b_{-1} - 2k a_0 a_{-1}^2 b_0 + 8k^3 a_1 b_{-1}^3,$$

$$C_{-3} = k a_0 a_{-1}^2 b_{-1} + \omega a_0 b_{-1}^3 - k a_{-1}^3 b_0 + k^3 a_0 b_{-1}^3 - \omega a_{-1} b_0 b_{-1}^2 - k^3 a_{-1} b_0 b_{-1}^2,$$

Equating the coefficients of $\exp(n\eta)$ to be zero, we have

$$C_3 = C_2 = C_1 = C_0 = C_{-1} = C_{-2} = C_{-3} = 0 \tag{1.1.14}$$

Solving the system, eq. (1.14), simultaneously, we obtain

$$\begin{cases} a_0 = a_1 b_0 + \dfrac{3k^2 b_0}{a_1}, \qquad a_{-1} = \dfrac{b_0^2(3k^2 + 2a_1^2)}{8a_1}, \quad b_{-1} = \dfrac{b_0^2(3k^2 + 2a_1^2)}{8a_1^2} \\ \omega = -k a_1^2 - k^3 \end{cases} \tag{1.1.15}$$

where a_1 and b_0 are free parameters.

We, therefore, obtain the following solutions:

$$u(x,t)=\frac{a_1\exp[kx-(ka_1^2+k^3)t]+a_1b_0+\left(3k^2b_0/a_1\right)+\left(b_0^2(3k^2+2a_1^2)/8a_1\right)\exp[-kx+(ka_1^2+k^3)t]}{\exp[kx-(ka_1^2+k^3)t]+b_0+\left(b_0^2(3k^2+2a_1^2)/8a_1^2\right)\exp[-kx+(ka_1^2+k^3)t]}$$

$$=a_1+\frac{\left(3k^2b_0/a_1\right)}{\exp[kx-(ka_1^2+k^3)t]+b_0+\left(b_0^2(3k^2+2a_1^2)/8a_1^2\right)\exp[-kx+(ka_1^2+k^3)t]} \qquad (1.1.16)$$

Generally a_1, b_0 and k are real numbers, and the obtained solution, eq. (1.1.16), is a generalized solitonary solution.

In case k is an imaginary number, the obtained solitionary solution can be converted into periodic solution or compact like solution. We write

$$k=iK. \qquad (1.1.17)$$

Using the transformation

$$\exp[kx-(ka_1^2+k^3)t]=\exp[iKx-i(Ka_1^2-K^3)t]$$

$$=\cos[Kx-(Ka_1^2-K^3)t]+i\sin[Kx-(Ka_1^2-K^3)t]$$

and

$$\exp[-kx+(ka_1^2+k^3)t]=\exp[-iKx+i(Ka_1^2-K^3)t]$$

$$=\cos[Kx-(Ka_1^2-K^3)t]-i\sin[Kx-(Ka_1^2-K^3)t]$$

eq. (1.1.16) becomes

$$u(x,t)=a_1+\frac{-\left(3k^2b_0 \big/ a_1\right)}{(1+p)\cos[Kx-(Ka_1^2-K^3)t]+b_0+i(1-p)\sin[Kx-(Ka_1^2-K^3)t]}, \tag{1.1.18}$$

where $p=\dfrac{b_0^2(-3K^2+2a_1^2)}{8a_1^2}$,

If we search for a periodic solution or compact-like solution, the imaginary part in the denominator of eq. (1.1.18) must be zero,

$$1-p=1-\frac{b_0^2(-3K^2+2a_1^2)}{8a_1^2}=0 \tag{1.1.19}$$

Solving b_0 from eq. (1.1.19) we obtain

$$b_0=\pm\sqrt{8\big/(-3K^2+2a_1^2)} \tag{1.1.20}$$

Substituting eq. (1.1.20) into eq. (1.1.18) results a compact-like

$$u(x,t)=a_1+\frac{\mp 3K^2\sqrt{8\big/(-3K^2+2a_1^2)}}{2\cos[Kx-(Ka_1^2-K^3)t]\pm\sqrt{8\big/\left(-3K^2+2a_1^2\right)}a_1}$$

$$=a_1+\frac{\mp 3K^2\sqrt{2\big/(-3K^2+2a_1^2)}}{\cos[Kx-(Ka_1^2-K^3)t]\pm\sqrt{2\big/(-3K^2+2a_1^2)}a_1} \tag{1.1.21}$$

where a_1 and K are free parameters and it requires that $2a_1^2>3K^2$.

So the suggested Exp-function method can obtain easily the generalized solitionary solution and compact-like solution for nonlinear equations.

CHAPTER 2

2.1 *Exp-function method for solving nonlinear evolution equations*

The two dimensional Bratu-type equation is given as:

$$u_{xx} + u_{yy} + \lambda \exp(su) = 0 \tag{2.1.1}$$

and the generalized Fisher equation with higher order nonlinearity is given as:

$$u_t = u_{xx} + u(1 - u^n) \tag{2.1.2}$$

Two dimensional Bratu model stimulates a thermal reaction process in a rigid material where the process depends on a balance between chemically generated heat addition and heat transfer by conduction. The nonlinear reaction-diffusion equation was first introduced by Fisher as a model for the propagation of a mutual gene. It has wide application in the fields of logistic population growth, flame propagation, neurophysiology, autocatalytic chemical reactions and nuclear reactor theory. It is well known that wave phenomena of plasma media and fluid dynamics are modelled by kink-shaped and $tanh$ solution or bell-shaped $sech$ solutions.

2.2 *Exp-Function method and application to two dimensional Bratu type equation*

Using a complex variation $\eta = kx + wy$ and the transformation $u = \left(\frac{1}{s}\right)\ln v$, we can convert eq. (2.1.1) into ordinary differential equation

$$\left(k^2 + w^2\right)vv'' - (k^2 + w^2)\left(v'\right)^2 + \lambda s v^3 = 0 \tag{2.2.1}$$

According to Exp-function method, we assume that the solution of eq. (2.2.1) can be expressed in the form

$$v(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} \tag{2.2.2}$$

where c , p d and q are positive integers, a_n and b_m are unknown constants.

For simplicity, we set $p=c=1$ and $q=d=1$, then eq. (2.2.2) reduces to

$$v(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)} \tag{2.2.3}$$

Putting eq. (2.2.3) into eq. (2.2.1), we get

$$\frac{1}{A}[C_4\exp(4\eta)+\cdots+C_1\exp(\eta)+C_0+C_{-1}\exp(-\eta)+\cdots+C_{-4}\exp(-4\eta)]=0 \tag{2.2.4}$$

Equating the coefficients of $\exp(n\eta)$ in eq. (2.2.4) to be zero yields a set of algebraic equations:

$$C_4=C_3=C_2=C_1=C_0=C_{-1}=C_{-2}=C_{-3}=C_{-4}=0$$

Solving the system of algebraic equations given above with the aid of symbolic computation system of mathematical software, we obtain;

$$a_{-1}=0,\ a_0=a_0,\ b_{-1}=\frac{b_0^2}{4b_1}\ ,\ b_0=b_0,\ w=\mp\sqrt{(\lambda s a_0-k^2 b_0)/b_0}, \tag{2.2.5}$$

Substituting eq. (2.2.5) into eq. (2.2.3), we obtain the following exact solution

$$v(x,y)=\frac{a_0}{\exp(kx+wy)+b_0+\left(b_0^2/4\right)\exp(-(kx+wy))} \tag{2.2.6}$$

So,

$$u(x,y)=\frac{1}{s}\ln\left(\frac{a_0}{\exp(kx+wy)+b_0+\left(b_0^2/4\right)\exp(-(kx+wy))}\right) \tag{2.2.7}$$

Using the properties

$$\exp(kx+wy)+\exp(-(kx+wy))=2\cosh(kx+wy) \tag{2.2.8}$$

When $b_0=\mp 2$, eq. (2.2.7) reduces to travelling wave solution as follows:

$$u(x,y)=\frac{1}{s}\ln\left(\frac{a_0}{2\cosh\left(kx\mp\sqrt{(\lambda s a_0-2k^2)/2}\,y\right)\mp 2}\right) \tag{2.2.9}$$

2.3 The generalized Fisher equation:

The generalized Fisher equation above in eq. (2.1.2) is

$$u_t=u_{xx}+u(1-u^n)$$

Introducing the complex variation η defined as $\eta=kx+wt$, we have

$$k^2u''-wu'-u^{n+1}+u=0 \tag{2.3.1}$$

where w and k are real parameter.

Making the transformation

$$u=v^{1/n} \tag{2.3.2}$$

eq. (2.3.1) becomes

$$k^2 n v v'' + k^2(1-n)(v')^2 - w n v v' - n^2 v^3 + n^2 v^2 = 0 \tag{2.3.3}$$

We assume that the solution of eq. (2.3.2) can be expressed in the form

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \tag{2.3.4}$$

Substituting eq. (2.3.4) into (2.3.3) and by the help of mathematical software, equating the coefficients of all powers of $\exp(n\eta)$ to be zero, $(n = -4, -3, \ldots, 3, 4)$ yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, k, w solving this system of algebraic equations by using mathematical software, we obtain the following results,

Case I.

$$a_{-1} = 0, a_0 = 0, a_1 = 1, b_{-1} = \frac{b_0^2}{4}, b_0 = b_0, k = \mp \frac{n}{\sqrt{2(n+2)}}, w = \frac{n(n+4)}{2(n+2)} \tag{2.3.5}$$

where b_0 is a free parameter, substituting these results into eq. (2.3.4), from (2.3.2) we obtain the following exact solution

$$u(x,t) = \left(\frac{\exp(kx + wt)}{\exp(kx + wt) + b_0 + \left(b_0^2 / 4 \right) \exp(-(kx + wt))} \right)^{\frac{1}{n}} \tag{2.3.6}$$

Using the properties

$$\exp(kx + wy) + \exp(-(kx + wy)) = 2\cosh(kx + wy) \tag{2.3.7}$$

$$\exp(kx + wy) - \exp(-(kx + wy)) = 2\sinh(kx + wy) \tag{2.3.8}$$

when $b_0 = \mp 2$, eq. (2.3.5) becomes,

$$u(x,t)=\left[\frac{\cosh(\mp A(x\mp Bt))+\sinh(\mp A(x\mp Bt))}{\mp 2+2\cosh(\mp A(x\mp Bt))}\right]^{\frac{1}{n}} \tag{2.3.9}$$

where $A=\dfrac{n}{\sqrt{2(n+2)}}$, $B=\dfrac{n+4}{\sqrt{2(n+2)}}$

Case II.

$$a_{-1}=b_{-1}, a_0=0, a_1=0, b_{-1}=\frac{b_0^2}{4}, b_0=b_0, k=\mp\frac{n}{\sqrt{2(n+2)}}, w=\frac{n(n+4)}{2(n+2)} \tag{2.3.10}$$

Where b_0 is a free parameter, substituting these results into eq. (2.3.4), and from (2.32) we obtain the following exact solution

$$u(x,t)=\left(\frac{\left(b_0^2/4\right)\exp(-(kx+wt))}{\exp(kx+wt)+b_0+\left(b_0^2/4\right)\exp(-(kx+wt))}\right)^{\frac{1}{n}} \tag{2.3.11}$$

Using the properties (2.3.7)-(2.3.8), when $b_0=\mp 2$ eq. (2.3.11) becomes

$$u(x,t)=\left[\frac{\cosh(\mp A(x\mp Bt))-\sinh(\mp A(x\mp Bt))}{\mp 2+2\cosh(\mp A(x\mp Bt))}\right]^{\frac{1}{n}} \tag{2.3.12}$$

where $A=\dfrac{n}{\sqrt{2(n+2)}}$, $B=\dfrac{n+4}{\sqrt{2(n+2)}}$

Conclusion

In this chapter, we applied Exp-function method for obtaining the exact solutions of the two dimensional Bratu-type equation and the generalized Fisher equation. The result show that this method is a powerful and effective mathematical tool for solving nonlinear evolution equation in science and engineering.

CHAPTER 3

Application of Exp-function to coupled Boussinesq-Burgers equations

A well-known model is the coupled Boussinesq-Burgers equations [2]

$$u_t = -2uu_x + \frac{1}{2}v_x, \tag{3.1}$$

$$v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \tag{3.2}$$

Using the transformation $u = u(\eta)$, $v = v(\eta)$, $\eta = kx + wt$ eqs. (3.1) and (3.2) become the ordinary differential equations,

$$wu' + 2kuu' - \frac{1}{2}kv' = 0, \tag{3.3}$$

$$wv' - \frac{1}{2}k^3u''' + 2k(uv)' = 0, \tag{3.4}$$

On integrating eqs. (3.3) and (3.4) once with reference to η and assuming that the constant of integration is zero, we get

$$wu + ku^2 - \frac{1}{2}kv = 0 \tag{3.5}$$

$$wv - \frac{1}{2}k^3u'' + 2kuv = 0 \tag{3.6}$$

According to Exp-function method, we assume that the solution of eqs. (3.3) and (3.4) can be expressed in the form,

$$u(\eta)=\frac{\sum_{n=-g}^{h} a_n \exp(n\eta)}{\sum_{m=-q}^{p} b_m \exp(m\eta)}=\frac{a_{-g}\exp(-g\eta)+\cdots+a_h\exp(h\eta)}{b_{-q}\exp(-q\eta)+\cdots+b_p\exp(p\eta)} \tag{3.7}$$

$$v(\eta)=\frac{\sum_{n=-i}^{j} d_n \exp(n\eta)}{\sum_{m=-s}^{t} c_m \exp(m\eta)}=\frac{d_{-i}\exp(-i\eta)+\cdots+d_j\exp(j\eta)}{c_{-s}\exp(-s\eta)+\cdots+c_t\exp(t\eta)} \tag{3.8}$$

Where h, g, p, q, j, i, t and s are positive integers a_n, b_m, d_n and c_m are unknown constants.

For simplicity, we set $h=p=1, g=q=1$ and $j=t=1, i=s=1$, then eqs. (3.7) and (3.8) reduced to

$$u(\eta)=\frac{a_{-1}\exp(-\eta)+a_0+a_1\exp(\eta)}{b_{-1}\exp(-\eta)+b_0+\exp(\eta)} \tag{3.9}$$

$$v(\eta)=\frac{d_{-1}\exp(-\eta)+d_0+d_1\exp(\eta)}{c_{-1}\exp(-\eta)+c_0+\exp(\eta)} \tag{3.10}$$

Substituting eqs. (3.9) and (3.10) into eqs. (3.5) and (3.6), we have

$$\frac{1}{A}\{\delta_{-3}\exp(-3\eta)+\delta_{-2}\exp(-2\eta)+\delta_{-1}\exp(-\eta)+\delta_0+\delta_1\exp(\eta)+\delta_2\exp(2\eta)$$

$$+\delta_3\exp(3\eta)\}=0 \tag{3.11}$$

$$\frac{1}{B}\{\xi_{-4}\exp(-4\eta)+\xi_{-3}\exp(-3\eta)+\xi_{-2}\exp(-2\eta)+\xi_{-1}\exp(-\eta)+\xi_0+\xi_1\exp(\eta)$$

$$+\xi_2\exp(2\eta)+\xi_3\exp(3\eta)+\xi_4\exp(4\eta)\}=0 \tag{3.12}$$

where

$$A=\left(b_{-1}\exp(-\eta)+b_0+\exp(\eta)\right)^2\left(c_{-1}\exp(-\eta)+c_0+\exp(\eta)\right)$$

$$B = \left(b_{-1}\exp(-\eta) + b_0 + \exp(\eta)\right)^3 \left(c_{-1}\exp(-\eta) + c_0 + \exp(\eta)\right)$$

$$\delta_{-3} = ka_{-1}^2 c_{-1} + wa_{-1}b_{-1}c_{-1} - \frac{1}{2}kb_{-1}^2 d_{-1},$$

$$\delta_{-2} = 2ka_{-1}a_0c_{-1} + wa_0b_{-1}c_{-1} + wa_{-1}b_0c_{-1} + ka_{-1}^2c_0 + wa_{-1}b_{-1}c_0 - kb_{-1}b_0d_{-1} - \frac{1}{2}kb_{-1}^2d_0,$$

$$\delta_{-1} = ka_{-1}^2 + wa_{-1}b_{-1} + wa_{-1}c_{-1} + ka_0^2c_{-1} + 2ka_{-1}a_1c_{-1} + wa_1b_{-1}c_{-1} + wa_0b_0c_{-1} + 2ka_{-1}a_0c_0$$

$$+ wa_0b_{-1}c_0 + wa_{-1}b_0c_0 - kb_{-1}d_{-1} - \frac{1}{2}kb_0^2d_{-1} - kb_{-1}b_0d_0 - \frac{1}{2}kb_{-1}^2d_1,$$

$$\delta_0 = \frac{1}{2}(4ka_{-1}a_0 + 2wa_0b_{-1} + 2wa_{-1}b_0 + 2wa_0c_{-1} + 4ka_0a_1c_{-1} + 2wa_1b_0c_{-1} + 2wa_{-1}c_0$$

$$+ 2ka_0^2c_0 + 4ka_{-1}a_1c_0 + 2wa_1b_{-1}c_0 + 2wa_0b_0c_0 - 2kb_0d_{-1} - 2kb_{-1}d_0 - kb_0^2d_0$$

$$- 2kb_{-1}b_0d_1),$$

$$\delta_1 = wa_{-1} + ka_0^2 + 2ka_{-1}a_1 + wa_1b_{-1} + wa_0b_0 + wa_1c_{-1} + ka_1^2c_{-1} + wa_0c_0 + 2ka_0a_1c_0$$

$$+ wa_1b_0c_0 - \frac{1}{2}kd_{-1} - kb_0d_0 - kb_{-1}d_1 - \frac{1}{2}kb_0^2d_1,$$

$$\delta_2 = wa_0 + 2ka_0a_1 + wa_1b_0 + wa_1c_0 + ka_1^2c_0 - \frac{1}{2}kd_0 - kb_0d_1,$$

$$\delta_3 = wa_1 + ka_1^2 - \frac{1}{2}kd_1,$$

$$\xi_{-4} = 2ka_{-1}b_{-1}^2d_{-1} + wb_{-1}^3d_{-1},$$

$$\xi_{-3} = -\frac{1}{2}k^3a_0b_{-1}^2c_{-1} + \frac{1}{2}k^3a_{-1}b_{-1}b_0c_{-1} + 2ka_0b_{-1}^2d_{-1} + 4ka_{-1}b_{-1}b_0d_{-1} + 3wb_{-1}^2b_0d_{-1}$$

$$+2ka_{-1}b_{-1}^2d_0+wb_{-1}^3d_0$$

$$\xi_{-2}=2k^3a_{-1}b_{-1}c_{-1}-2k^3a_1b_{-1}^2c_{-1}+\frac{1}{2}k^3a_0b_{-1}b_0c_{-1}-\frac{1}{2}k^3a_{-1}b_0^2c_{-1}-\frac{1}{2}k^3a_0b_{-1}^2c_0+$$

$$\frac{1}{2}k^3a_{-1}b_{-1}b_0c_0+4ka_{-1}b_{-1}d_{-1}+3wb_{-1}^2d_{-1}+2ka_1b_{-1}^2d_{-1}+4ka_0b_{-1}b_0d_{-1}+$$

$$2ka_{-1}b_0^2d_{-1}+3wb_{-1}b_0^2d_{-1}+2ka_0b_{-1}^2d_0+4ka_{-1}b_{-1}b_0d_0+3wb_{-1}^2b_0d_0+$$

$$2ka_{-1}b_{-1}^2d_1+wb_{-1}^3d_1,$$

$$\xi_{-1}=-\frac{1}{2}k^3a_0b_{-1}^2+\frac{1}{2}k^3a_{-1}b_{-1}b_0+3k^3a_0b_{-1}c_{-1}-\frac{3}{2}k^3a_{-1}b_0c_{-1}-\frac{3}{2}k^3a_1b_{-1}b_0c_{-1}+$$

$$2k^3a_{-1}b_{-1}c_0-2k^3a_1b_{-1}^2c_0+\frac{1}{2}k^3a_0b_{-1}b_0c_0-\frac{1}{2}k^3a_{-1}b_0^2c_0+4ka_0b_{-1}d_{-1}+$$

$$4ka_{-1}b_0d_{-1}+6wb_{-1}b_0d_{-1}+4ka_1b_{-1}b_0d_{-1}+2ka_0b_0^2d_{-1}+wb_0^3d_{-1}+4ka_{-1}b_{-1}d_0$$

$$+3wb_{-1}^2d_0+2ka_1b_{-1}^2d_0+4ka_0b_{-1}b_0d_0+2ka_{-1}b_0^2d_0+3wb_{-1}b_0^2d_0+2ka_0b_{-1}^2d_1$$

$$+4ka_{-1}b_{-1}b_0d_1+3wb_{-1}^2b_0d_1,$$

$$\xi_0=\frac{1}{2}(4k^3a_{-1}b_{-1}-4k^3a_1b_{-1}^2+k^3a_0b_{-1}b_0-k^3a_{-1}b_0^2-4k^3a_{-1}c_{-1}+4k^3a_1b_{-1}c_{-1}+$$

$$k^3a_0b_0c_{-1}-k^3a_1b_0^2c_{-1}+6k^3a_0b_{-1}c_0-3k^3a_{-1}b_0c_0-3k^3a_1b_{-1}b_0c_0+4ka_{-1}d_{-1}+$$

$$6wb_{-1}d_{-1}+8ka_1b_{-1}d_{-1}+8ka_0b_0d_{-1}+6wb_0^2d_{-1}+4ka_1b_0^2d_{-1}+8ka_0b_{-1}d_0+$$

$$8ka_{-1}b_0d_0+12wb_{-1}b_0d_0+8ka_1b_{-1}b_0d_0+4ka_0b_0^2d_0+2wb_0^3d_0+8ka_{-1}b_{-1}d_1$$

$$+6wb_{-1}^2d_1+4ka_1b_{-1}^2d_1+8ka_0b_{-1}b_0d_1+4ka_{-1}b_0^2d_1+6wb_{-1}b_0^2d_1)$$

$$\xi_1 = 3k^3a_0b_{-1} - \frac{3}{2}k^3a_{-1}b_0 - \frac{3}{2}k^3a_1b_{-1}b_0 - \frac{1}{2}k^3a_0c_{-1} + \frac{1}{2}k^3a_1b_0c_{-1} - 2k^3a_{-1}c_0 +$$

$$2k^3a_1b_{-1}c_0 + \frac{1}{2}k^3a_0b_0c_0 - \frac{1}{2}k^3a_1b_0^2c_0 + 2ka_0d_{-1} + 3wb_0d_{-1} + 4ka_1b_0d_{-1} +$$

$$2ka_{-1}d_0 + 3wb_{-1}d_0 + 4ka_1b_{-1}d_0 + 4ka_0b_0d_0 + 3wb_0^3d_0 + 2ka_1b_0^2d_0 + 4ka_0b_{-1}d_1$$

$$+ 4ka_{-1}b_0d_1 + 6wb_{-1}b_0d_1 + 4ka_1b_{-1}b_0d_1 + 2ka_0b_0^2d_1 + wb_0^3d_1$$

$$\xi_2 = -2k^3a_{-1} + 2k^3a_1b_{-1} + \frac{1}{2}k^3a_0b_0 - \frac{1}{2}k^3a_1b_0^2 - \frac{1}{2}k^3a_0c_0 + \frac{1}{2}k^3a_1b_0c_0 + wd_{-1} +$$

$$2ka_1d_{-1} + 2ka_0d_0 + 3wb_0d_0 + 4ka_1b_0d_0 + 2ka_{-1}d_1 + 3wb_{-1}d_1 + 4ka_1b_{-1}d_1$$

$$+ 4ka_0b_0d_1 + 3wb_0^2d_1 + 2ka_1b_0^2d_1$$

$$\xi_3 = -\frac{1}{2}k^3a_0 + \frac{1}{2}k^3a_1b_0 + wd_0 + 2ka_1d_0 + 2ka_0d_1 + 3wb_0d_1 + 4ka_1b_0d_1$$

$$\xi_4 = wd_1 + 2ka_1d_1$$

By setting,

$\delta_{-3} = \delta_{-2} = \delta_{-1} = \delta_0 = \delta_1 = \delta_2 = \delta_3 = 0$, and

$\xi_{-4} = \xi_{-3} = \xi_{-2} = \xi_{-1} = \xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$,

and solving these system of algebraic equations simultaneously with the aid of symbolic computation system of mathematical software, we obtain the following results,

Case I.

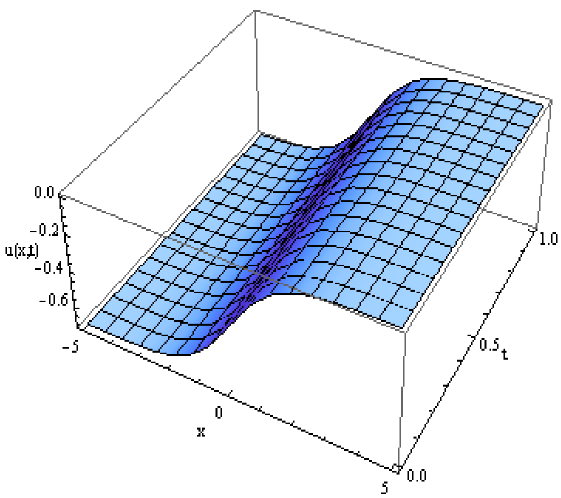
$$a_{-1}=0, a_0=-\frac{\sqrt{w}b_0}{\sqrt{2}}, a_1=0, b_{-1}=0, d_{-1}=0, d_0=-wb_0, d_1=0, c_{-1}=b_0^2,$$

$$c_0=2b_0, k=\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t)=-\frac{\sqrt{w}b_0}{\sqrt{2}[b_0+\cosh(kx+wt)+\sinh(kx+wt)]},$$

$$v(x,t)=-\frac{wb_0}{2b_0+\cosh(kx+wt)+b_0^2\left(\cosh(kx+wt)-\sinh(kx+wt)\right)+\sinh(kx+wt)},$$



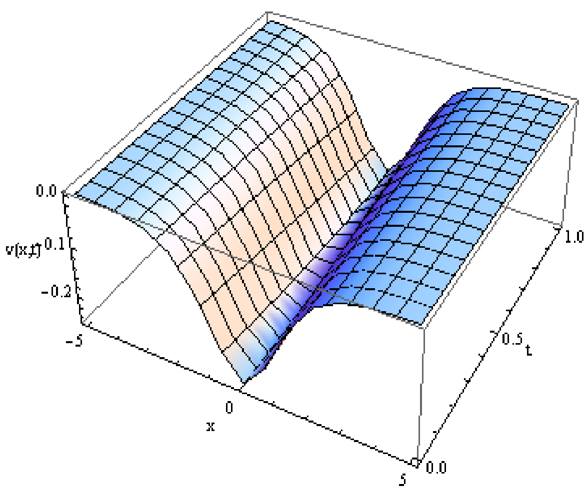


Fig. 3(a). Soliton solutions of eqs. (3.9) and (3.10) in case I, when $b_0=w=1$.

Case II.

$$a_{-1}=0, a_0=\frac{\sqrt{w}b_0}{\sqrt{2}}, a_1=0, b_{-1}=0, d_{-1}=0, d_0=-wb_0, d_1=0, c_{-1}=b_0^2,$$

$$c_0=2b_0, k=-\sqrt{2w}$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t)=\frac{\sqrt{w}b_0}{\sqrt{2}\left(b_0+\cosh(kx+wt)+\sinh(kx+wt)\right)},$$

$$v(x,t)=-\frac{wb_0}{2b_0+\cosh(kx+wt)+b_0^2\left(\cosh(kx+wt)-\sinh(kx+wt)\right)+\sinh(kx+wt)}$$

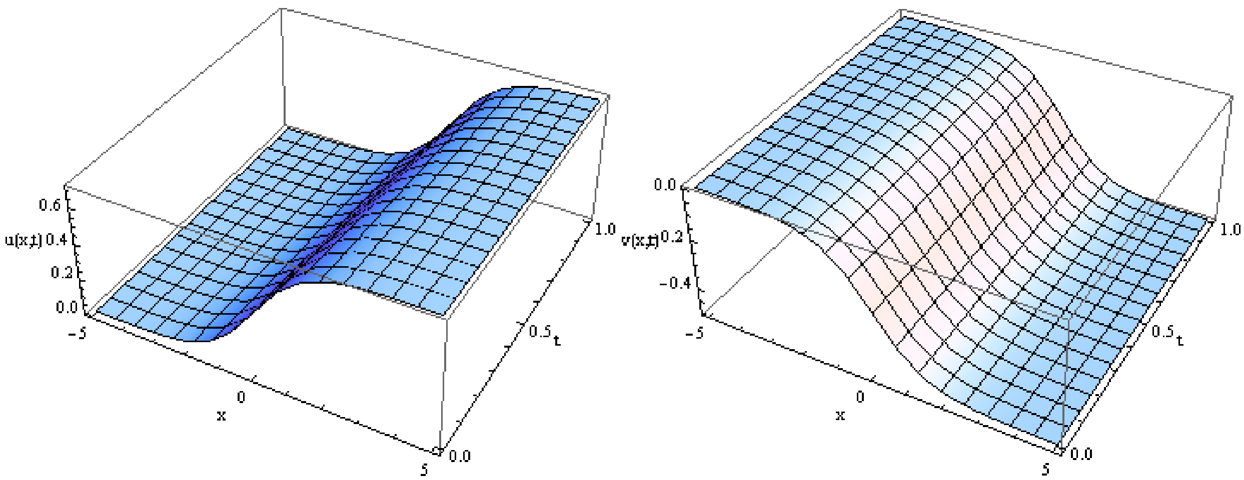


Fig. 3(b). Soliton solutions of eqs. (3.9) and (3.10) in case II, when $b_0=w=1$.

Case III.

$$a_{-1}=0, a_0=0, a_1=-\frac{\sqrt{w}}{\sqrt{2}}, b_{-1}=0, d_{-1}=0, d_0=-wb_0, d_1=0, c_{-1}=b_0^2,$$

$$c_0=2b_0, k=\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t)=-\frac{\sqrt{w}\left(\cosh(kx+wt)+\sinh(kx+wt)\right)}{\sqrt{2}\left(b_0+\cosh(kx+wt)+\sinh(kx+wt)\right)},$$

$$v(x,t)=-\frac{wb_0}{2b_0+\cosh(kx+wt)+b_0^2\left(\cosh(kx+wt)-\sinh(kx+wt)\right)+\sinh(kx+wt)},$$

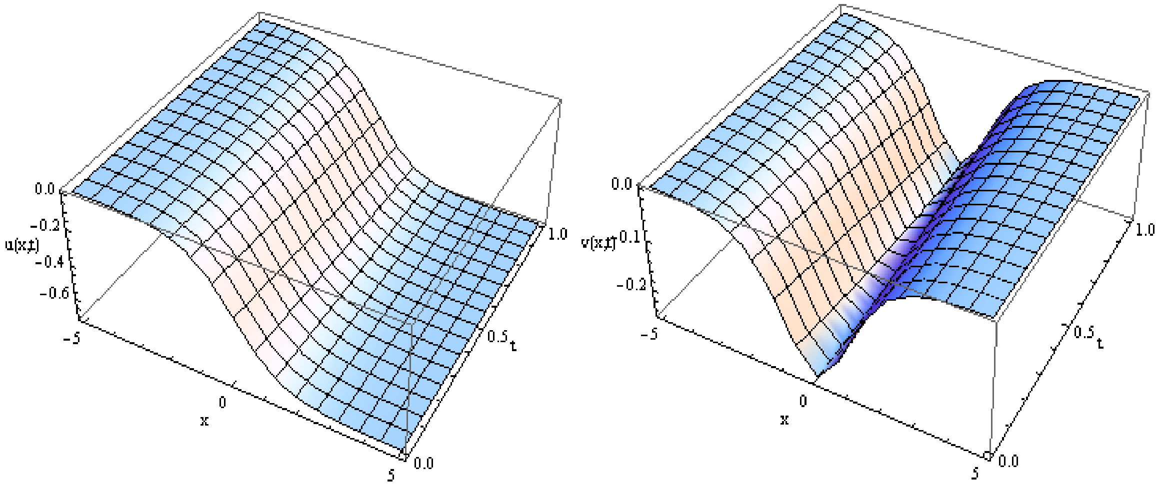


Fig. 3(c). Soliton solutions of eqs. (3.9) and (3.10) in case III, when $b_0 = w = 1$.

Case IV.

$$a_{-1} = 0, a_0 = 0, a_1 = \frac{\sqrt{w}}{\sqrt{2}}, b_{-1} = 0, d_{-1} = 0, d_0 = -wb_0, d_1 = 0, c_{-1} = b_0^2,$$

$$c_0 = 2b_0, k = -\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = \frac{\sqrt{w}\left(\cosh(kx+wt) + \sinh(kx+wt)\right)}{\sqrt{2}\left(b_0 + \cosh(kx+wt) + \sinh(kx+wt)\right)},$$

$$v(x,t) = -\frac{wb_0}{2b_0 + \cosh(kx+wt) + b_0^2\left(\cosh(kx+wt) - \sinh(kx+wt)\right) + \sinh(kx+wt)}$$

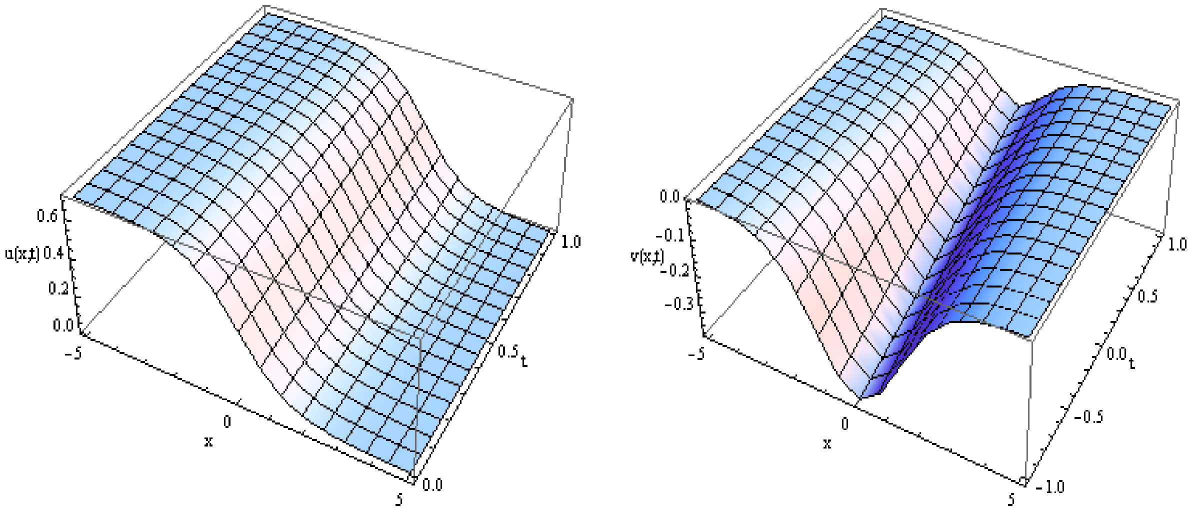


Fig. 3(d). Soliton solutions of eqs. (3.9) and (3.10) in case IV, when $b_0 = w = 1$.

Case V.

$$a_{-1} = \frac{\sqrt{w}b_{-1}}{\sqrt{2}}, a_0 = \frac{1}{4}[\sqrt{2w}b_0 - \sqrt{2(-4wb_{-1} + wb_0^2)}], a_1 = 0, d_{-1} = 0,$$

$$d_0 = \frac{-w^2 b_0 + w^{3/2}\sqrt{w(-4b_{-1} + b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{[3w^{5/2}b_{-1}b_0 - w^{5/2}b_0^3 - w^2(b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)})}{-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2[2w^{5/2}b_{-1} - w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1} - b_0^2)}]}{-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}}, k = -\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

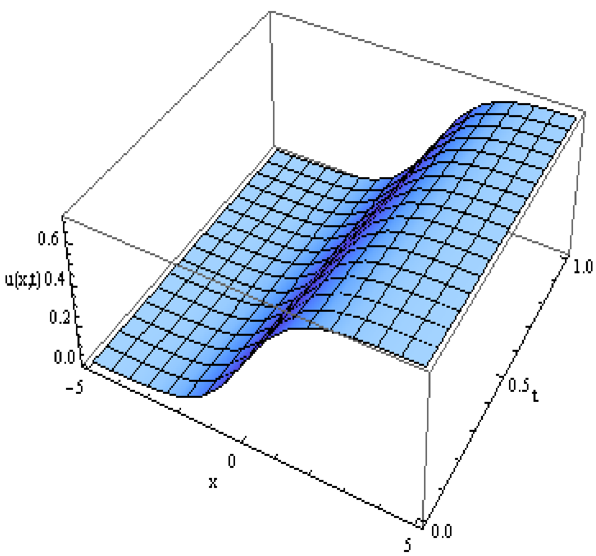
$$u(x,t) = \frac{\left[2\sqrt{w}b_{-1} + (\sqrt{w}b_0 - \sqrt{-4wb_{-1} + wb_0^2})(\sinh(kx+wt) + \cosh(kx+wt))\right]}{2\sqrt{2}\left(b_{-1} + b_0(\cosh(kx+wt) + \sinh(kx+wt)) + \cosh(2kx+2wt) + \sinh(2kx+2wt)\right)},$$

$$v(x,t)=-\left[\sqrt{w}\{-4wb_{-1}+2wb_0^2-2\sqrt{w}b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\}\right]\times$$

$$\frac{1}{\left(2\left(-3\sqrt{w}b_{-1}b_0+\sqrt{w}b_0^3+(b_{-1}-b_0^2)\sqrt{-4wb_{-1}+wb_0^2}+\left(-4\sqrt{w}b_{-1}+2\sqrt{w}b_0^2-\right.\right.\right.}$$

$$\left.2b_0\sqrt{-4wb_{-1}+wb_0^2}\right)\{\cosh(kx+wt)+\sinh(kx+wt)\}+\{\sqrt{w}b_0-$$

$$\left.\left.\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}\right)\right)$$



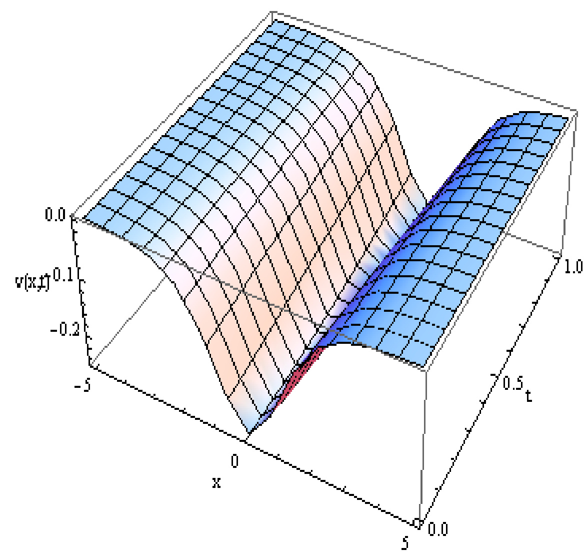


Fig. 3(e). Soliton solutions of eqs. (3.9) and (3.10) in case V, when $b_0=2, b_{-1}=1, w=1$.

Case VI.

$$a_{-1}=-\frac{\sqrt{w}b_{-1}}{\sqrt{2}}, a_0=\frac{1}{4}[-\sqrt{2w}b_0-\sqrt{2(-4wb_{-1}+wb_0^2)}], a_1=0, d_{-1}=0,$$

$$d_0=\frac{-w^2b_0-w^{3/2}\sqrt{w(-4b_{-1}+b_0^2)}}{2w}, d_1=0,$$

$$c_{-1}=\frac{\left[-3w^{5/2}b_{-1}b_0+w^{5/2}b_0^3-w^2(b_{-1}-b_0^2)\sqrt{-w(4b_{-1}-b_0^2)}\right]}{w^{5/2}b_0+w^2\sqrt{w(-4b_{-1}+b_0^2)}},$$

$$c_0=\frac{2\left[-2w^{5/2}b_{-1}+w^{5/2}b_0^2+w^2b_0\sqrt{-w(4b_{-1}-b_0^2)}\right]}{\left[w^{5/2}b_0+w^2\sqrt{w(-4b_{-1}+b_0^2)}\right]},k=\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

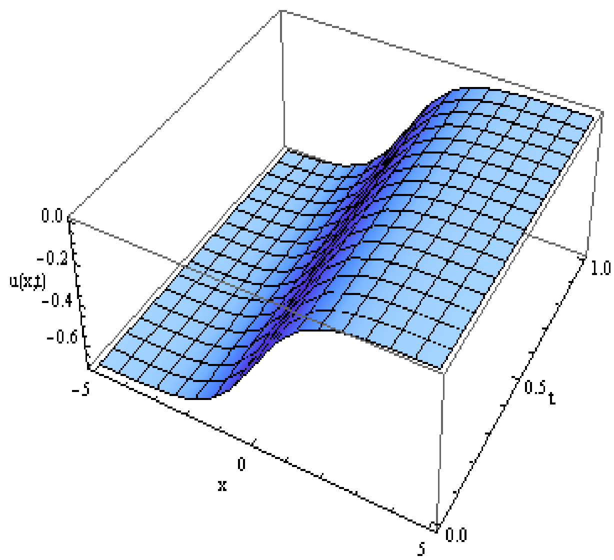
$$u(x,t)=\frac{\left[-2\sqrt{w}b_{-1}-(\sqrt{w}b_0+\sqrt{-4wb_{-1}+wb_0^2})\left(\sinh(kx+wt)+\cosh(kx+wt)\right)\right]}{2\sqrt{2}\left[b_{-1}+b_0\left(\cosh(kx+wt)+\sinh(kx+wt)\right)+\cosh(2kx+2wt)+\sinh(2kx+2wt)\right]},$$

$$v(x,t)=-\left[\sqrt{w}\left(-4wb_{-1}+2wb_0^2+2\sqrt{w}b_0\sqrt{-4wb_{-1}+wb_0^2}\right)\left(\cosh(kx+wt)+\sinh(kx+wt)\right)\right]\times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0+\sqrt{w}b_0^3+(-b_{-1}+b_0^2)\sqrt{-4wb_{-1}+wb_0^2}+\left(-4\sqrt{w}b_{-1}+2\sqrt{w}b_0^2+\right.\right.\right.}$$

$$\left.2b_0\sqrt{-4wb_{-1}+wb_0^2}\right)\{\cosh(kx+wt)+\sinh(kx+wt)\}+\left(\sqrt{w}b_0+\right.$$

$$\left.\left.\left.\sqrt{-4wb_{-1}+wb_0^2}\right)\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}\right)\right]$$



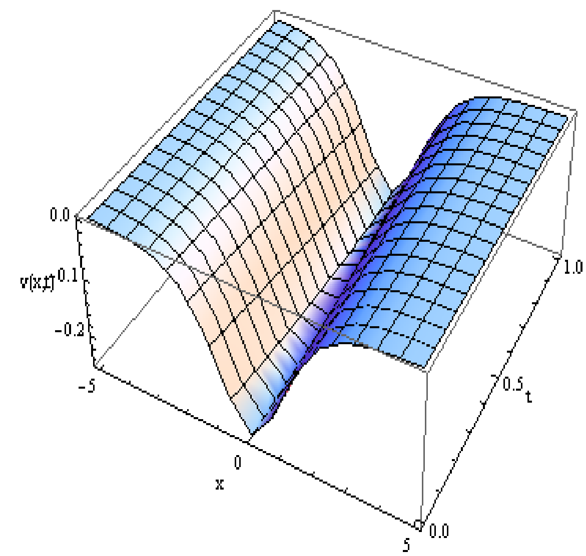


Fig. 3(f). Soliton solutions of eqs. (3.9) and (3.10) in case VI, when $b_0 = 2, b_{-1} = 1, w = 1$.

Case VII.

$$a_{-1} = \frac{\sqrt{w}b_{-1}}{\sqrt{2}}, a_0 = \frac{1}{4}[\sqrt{2w}b_0 + \sqrt{2(-4wb_{-1} + wb_0^2)}], a_1 = 0, d_{-1} = 0,$$

$$d_0 = \frac{-w^2 b_0 - w^{3/2}\sqrt{w(-4b_{-1} + b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{\left[-3w^{5/2}b_{-1}b_0 + w^{5/2}b_0^3 - w^2(b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)}\right]}{w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2\left[-2w^{5/2}b_{-1} + w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1} - b_0^2)}\right]}{\left[w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}\right]}, k = -\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = \frac{\left[2\sqrt{w}b_{-1} + \left(\sqrt{w}b_0 + \sqrt{-4wb_{-1} + wb_0^2}\right)(\sinh(kx + wt) + \cosh(kx + wt))\right]}{2\sqrt{2}\left[b_{-1} + b_0\{\cosh(kx + wt) + \sinh(kx + wt)\} + \cosh(2kx + 2wt) + \sinh(2kx + 2wt)\right]},$$

$$v(x,t) = -\left[\sqrt{w}\{-4wb_{-1} + 2wb_0^2 + 2\sqrt{w}b_0\sqrt{-4wb_{-1} + wb_0^2}\}\{Cosh(kx+wt) + Sinh(kx+wt)\}\right] \times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0 + \sqrt{w}b_0^3 + (-b_{-1} + b_0^2)\sqrt{-4wb_{-1} + wb_0^2} + (-4\sqrt{w}b_{-1} + 2\sqrt{w}b_0^2 + \right.\right.}$$

$$2b_0\sqrt{-4wb_{-1} + wb_0^2})(\cosh(kx+wt) + \sinh(kx+wt)) + (\sqrt{w}b_0 +$$

$$\left.\left.\sqrt{-4wb_{-1} + wb_0^2})(\cosh(2kx+2wt) + \sinh(2kx+2wt))\right)\right]$$

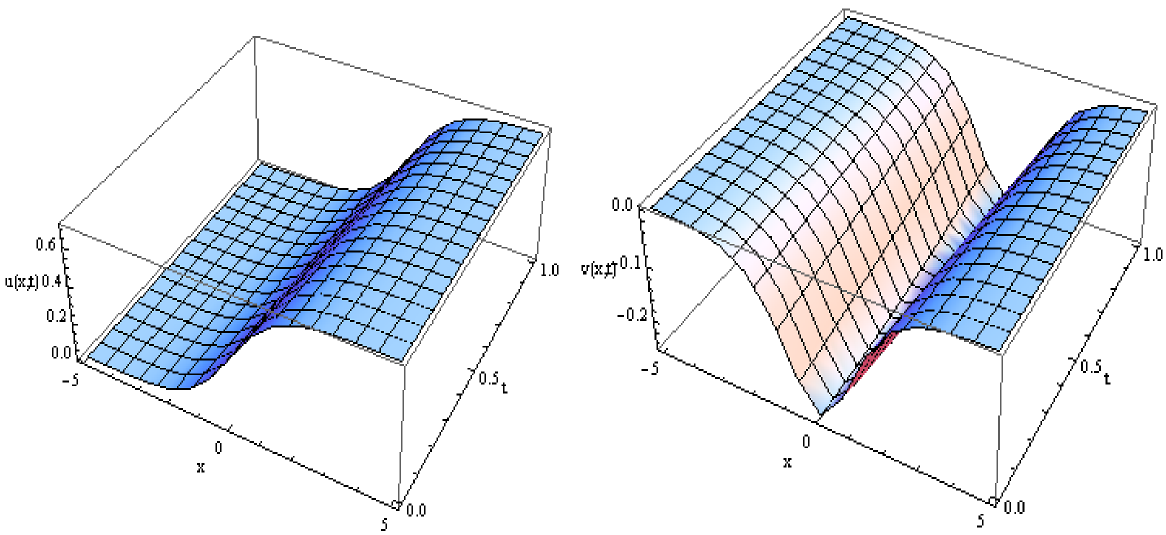


Fig. 3(g). Soliton solutions of eqs. (3.9) and (3.10) in case VII, when $b_0 = 0, b_{-1} = -1, w = 1$.

Case VIII.

$$a_{-1} = -\frac{\sqrt{w}b_{-1}}{\sqrt{2}}, a_0 = \frac{1}{4}[-\sqrt{2w}b_0 + \sqrt{2(-4wb_{-1} + wb_0^2)}], a_1 = 0, d_{-1} = 0,$$

$$d_0 = \frac{-w^2b_0 + w^{3/2}\sqrt{w(-4b_{-1} + b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{\left[3w^{5/2}b_{-1}b_0 - w^{5/2}b_0^3 - w^2(b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)}\right]}{-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2\left[2w^{5/2}b_{-1} - w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1} - b_0^2)}\right]}{\left[-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}\right]}, k = \sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = \frac{\left[-2\sqrt{w}b_{-1} - \left(\sqrt{w}b_0 - \sqrt{-4wb_{-1} + wb_0^2}\right)\left(\sinh(kx+wt) + \cosh(kx+wt)\right)\right]}{2\sqrt{2}\left[b_{-1} + b_0\left(\cosh(kx+wt) + \sinh(kx+wt)\right) + \cosh(2kx+2wt) + \sinh(2kx+2wt)\right]}$$

$$v(x,t) = -\left[\sqrt{w} \times \left(-4wb_{-1} + 2wb_0^2 - 2\sqrt{w}b_0\sqrt{-4wb_{-1} + wb_0^2}\right) \times \left(\cosh(kx+wt) + \sinh(kx+wt)\right)\right] \times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0 + \sqrt{w}b_0^3 + (b_{-1} - b_0^2)\sqrt{-4wb_{-1} + wb_0^2} + (-4\sqrt{w}b_{-1} + 2\sqrt{w}b_0^2 - \right.\right.}$$

$$2b_0\sqrt{-4wb_{-1} + wb_0^2})(\cosh(kx+wt) + \sinh(kx+wt)) + (\sqrt{w}b_0 -$$

$$\left.\left.\sqrt{-4wb_{-1} + wb_0^2})(\cosh(2kx+2wt) + \sinh(2kx+2wt))\right)\right]$$

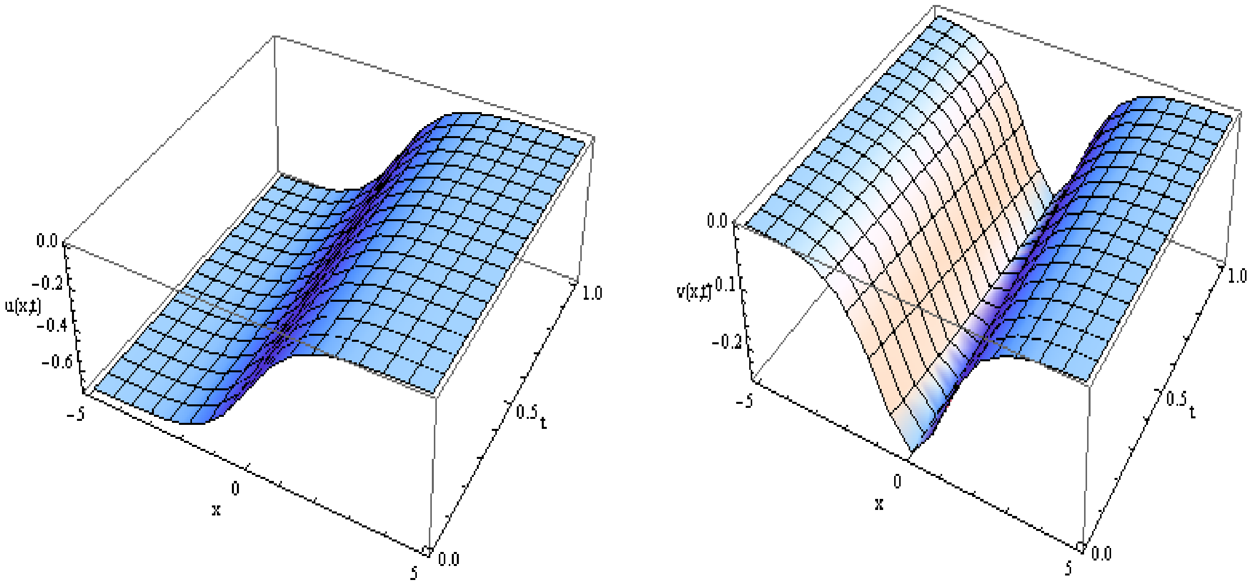


Fig. 3(h). Soliton solutions of eqs. (3.9) and (3.10) in case VIII, when $b_0 = 2, b_{-1} = 1, w = 1$.

Case IX.

$$a_{-1} = 0, a_0 = \frac{1}{4}(\sqrt{2w}b_0 + \sqrt{2(-4wb_{-1} + wb_0^2)}), a_1 = \frac{\sqrt{w}}{\sqrt{2}}, d_{-1} = 0,$$

$$d_0 = \frac{-w^2 b_0 + w^{3/2}\sqrt{w(-4b_{-1} + b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{(3w^{5/2}b_{-1}b_0 - w^{5/2}b_0^3 - w^2(b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)})}{-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2(2w^{5/2}b_{-1} - w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1} - b_0^2)})}{(-w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)})}, k = -\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = \Big[\{\cosh(kx + wt) + \sinh(kx + wt)\}\{\sqrt{w}b_0 + \sqrt{-4wb_{-1} + wb_0^2}\} + 2\sqrt{w}\{\cosh(kx + wt)$$

$$+\sinh(kx+wt)\}]\times\frac{1}{\left(2\sqrt{2}[b_{-1}+b_0\{\cosh(kx+wt)+\sinh(kx+wt)\}+\right.}$$

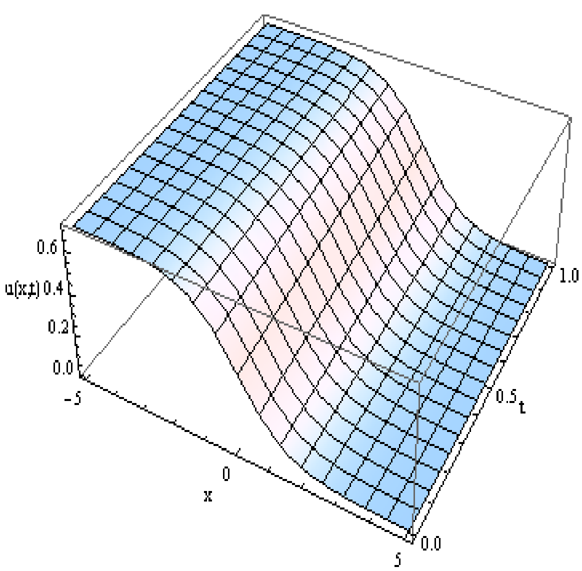
$$\left.\cosh(2kx+2wt)+\sinh(2kx+2wt)]\right)$$

$$v(x,t)=-\left[\sqrt{w}\left(-4wb_{-1}+2wb_0^2-2\sqrt{w}b_0\sqrt{-4wb_{-1}+wb_0^2}\right)\times\left(\cosh(kx+wt)+\sinh(kx+wt)\right)\right]\times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0+\sqrt{w}b_0^3+(b_{-1}-b_0^2)\sqrt{-4wb_{-1}+wb_0^2}+\{-4\sqrt{w}b_{-1}+2\sqrt{w}b_0^2-\right.\right.}$$

$$2b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\}+\{\sqrt{w}b_0-\sqrt{-4wb_{-1}+wb_0^2}\}$$

$$\left.\left.\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}\right)\right]$$



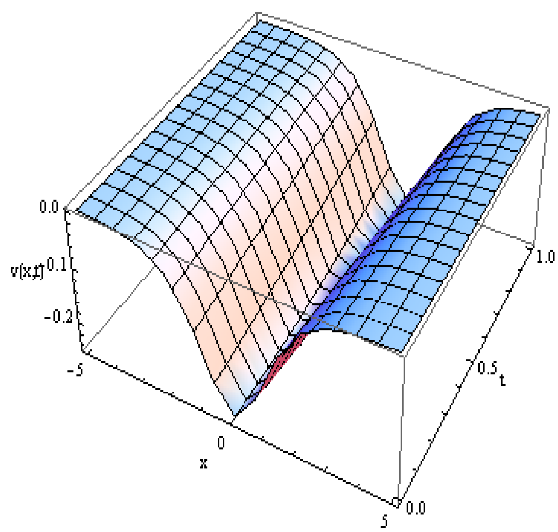


Fig. 3(i). Soliton solutions of eqs. (3.9) and (3.10) in case IX, when $b_0=2, b_{-1}=1, w=1$.

Case X.

$$a_{-1}=0, a_0=\frac{1}{4}(-\sqrt{2w}b_0-\sqrt{2(-4wb_{-1}+wb_0^2)}), a_1=-\frac{\sqrt{w}}{\sqrt{2}}, d_{-1}=0,$$

$$d_0 = \frac{-w^2 b_0 + w^{3/2}\sqrt{w(-4b_{-1}+b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{(3w^{5/2} b_{-1} b_0 - w^{5/2} b_0^3 - w^2 (b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)})}{-w^{5/2} b_0 + w^2 \sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2(2w^{5/2} b_{-1} - w^{5/2} b_0^2 + w^2 b_0 \sqrt{-w(4b_{-1} - b_0^2)})}{(-w^{5/2} b_0 + w^2 \sqrt{w(-4b_{-1} + b_0^2)})}, k = \sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = -[\{\cosh(kx+wt) + \sinh(kx+wt)\}\{\sqrt{w}b_0 + \sqrt{-4wb_{-1} + wb_0^2} + 2\sqrt{w}\{\cosh(kx+wt)$$

$$+\sinh(kx+wt)\}\}]\times \frac{1}{\left(2\sqrt{2}[b_{-1} + b_0\{\cosh(kx+wt) + \sinh(kx+wt)\} + \right.}$$

$$\left.\cosh(2kx+2wt) + \sinh(2kx+2wt)]\right)$$

$$v(x,t) = -\left[\sqrt{w}\{-4wb_{-1} + 2wb_0^2 - 2\sqrt{w}b_0\sqrt{-4wb_{-1} + wb_0^2}\}\{\cosh(kx+wt) + \sinh(kx+wt)\}\right]\times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0 + \sqrt{w}b_0^3 + (b_{-1} - b_0^2)\sqrt{-4wb_{-1} + wb_0^2} + \{-4\sqrt{w}b_{-1} + 2\sqrt{w}b_0^2 - \right.\right.}$$

$$2b_0\sqrt{-4wb_{-1} + wb_0^2}\}\{\cosh(kx+wt) + \sinh(kx+wt)\} + \{\sqrt{w}b_0 - \sqrt{-4wb_{-1} + wb_0^2}\}$$

$$\left.\left.\{\cosh(2kx+2wt) + \sinh(2kx+2wt)\}\right)\right],$$

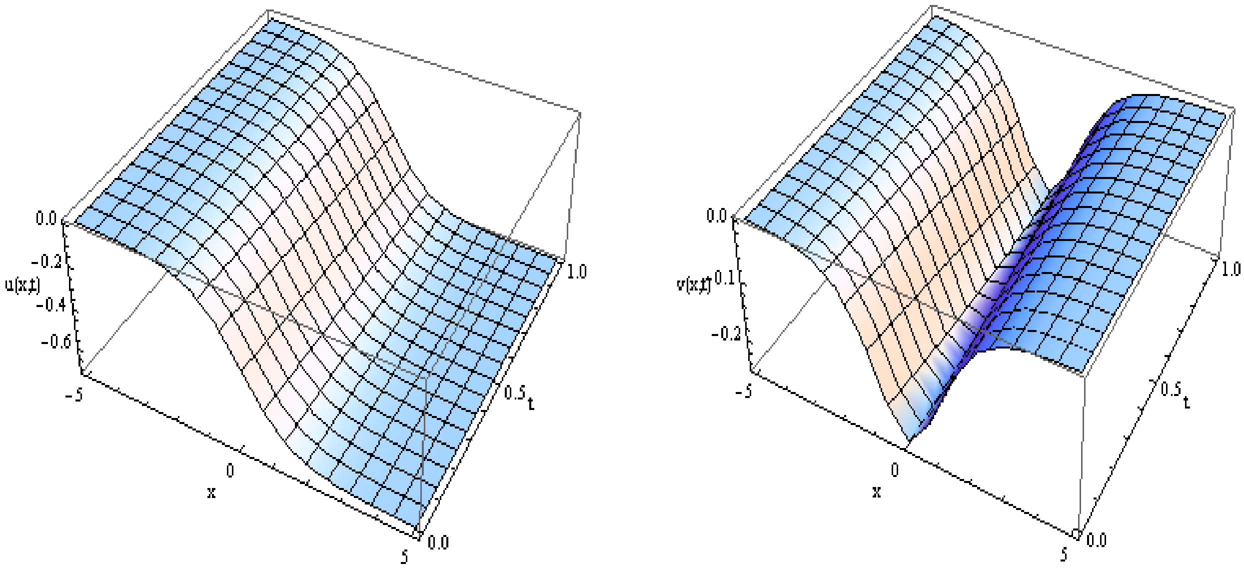


Fig. 3(j). Soliton solutions of eqs. (3.9) and (3.10) in case X, when $b_0 = 2, b_{-1} = 1, w = 1$.

Case XI.

$$a_{-1} = 0, a_0 = \frac{1}{4}(\sqrt{2w}b_0 - \sqrt{2(-4wb_{-1} + wb_0^2)}), a_1 = \frac{\sqrt{w}}{\sqrt{2}}, d_{-1} = 0,$$

$$d_0 = \frac{-w^2 b_0 - w^{3/2}\sqrt{w(-4b_{-1} + b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{\left[-3w^{5/2}b_{-1}b_0 + w^{5/2}b_0^3 - w^2(b_{-1} - b_0^2)\sqrt{-w(4b_{-1} - b_0^2)}\right]}{w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}},$$

$$c_0 = \frac{2\left[-2w^{5/2}b_{-1} + w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1} - b_0^2)}\right]}{\left[w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1} + b_0^2)}\right]}, k = -\sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = \Big[\{\cosh(kx + wt) + \sinh(kx + wt)\}\{\sqrt{w}b_0 - \sqrt{-4wb_{-1} + wb_0^2}\} + 2\sqrt{w}\{\cosh(kx + wt)$$

$$+\sinh(kx+wt)\}\Big]\times\frac{1}{\Big[2\sqrt{2}\big(b_{-1}+b_0\{\cosh(kx+wt)+\sinh(kx+wt)\}+}$$

$$\cosh(2kx+2wt)+\sinh(2kx+2wt)\big)\Big]$$

$$v(x,t)=-\Big[\sqrt{w}\{-4wb_{-1}+2wb_0^2+2\sqrt{w}b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\}\Big]\times$$

$$\frac{1}{\Big[\Big(2\{-3\sqrt{w}b_{-1}b_0+\sqrt{w}b_0^3+(-b_{-1}+b_0^2)\sqrt{-4wb_{-1}+wb_0^2}+\{-4\sqrt{w}b_{-1}+2\sqrt{w}b_0^2+}$$

$$2b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\}+\{\sqrt{w}b_0+\sqrt{-4wb_{-1}+wb_0^2}\}$$

$$\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}\Big)\Big],$$

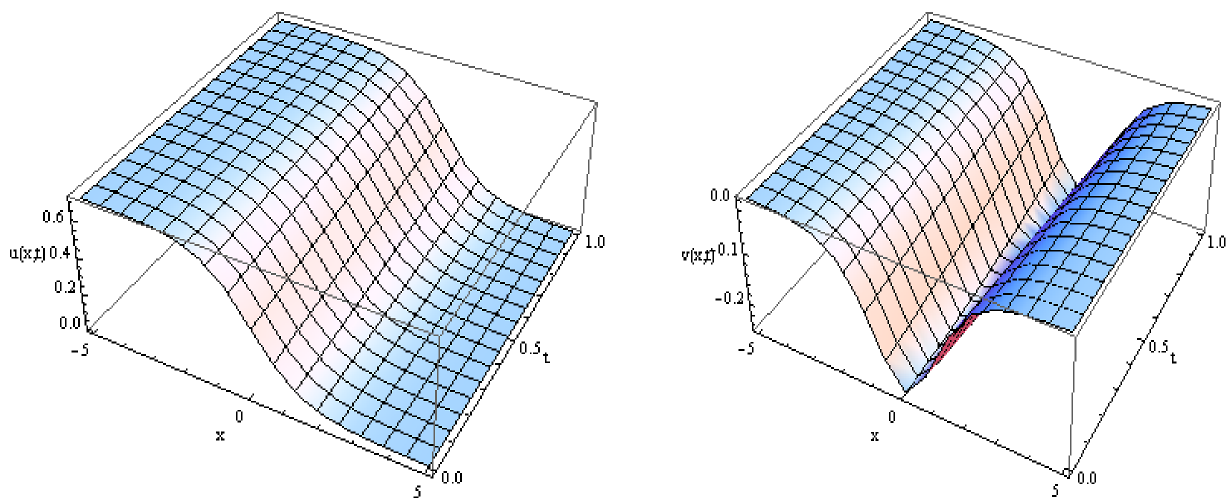


Fig. 3(k). Soliton solutions of eqs. (3.9) and (3.10) in case XI, when $b_0=0, b_{-1}=-1, w=1$

Case XII.

$$a_{-1}=0, a_0=\frac{1}{4}(-\sqrt{2w}b_0+\sqrt{2(-4wb_{-1}+wb_0^2)}), a_1=-\frac{\sqrt{w}}{\sqrt{2}}, d_{-1}=0,$$

$$d_0 = \frac{-w^2 b_0 - w^{3/2}\sqrt{w(-4b_{-1}+b_0^2)}}{2w}, d_1 = 0,$$

$$c_{-1} = \frac{\left[-3w^{5/2}b_{-1}b_0 + w^{5/2}b_0^3 - w^2(b_{-1}-b_0^2)\sqrt{-w(4b_{-1}-b_0^2)}\right]}{w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1}+b_0^2)}},$$

$$c_0 = \frac{2\left[-2w^{5/2}b_{-1} + w^{5/2}b_0^2 + w^2 b_0\sqrt{-w(4b_{-1}-b_0^2)}\right]}{(w^{5/2}b_0 + w^2\sqrt{w(-4b_{-1}+b_0^2)})}, k = \sqrt{2w},$$

Substituting these values into eqs. (3.9) and (3.10), we get:

$$u(x,t) = [\{\cosh(kx+wt)+\sinh(kx+wt)\}\{-\sqrt{w}b_0 + \sqrt{-4wb_{-1}+wb_0^2} - 2\sqrt{w}\{\cosh(kx+wt)$$

$$+\sinh(kx+wt)\}\}]\times\frac{1}{\left(2\sqrt{2}[b_{-1}+b_0\{\cosh(kx+wt)+\sinh(kx+wt)\}+\right.}$$

$$\left.\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}]\right)$$

$$v(x,t) = -\left[\sqrt{w}\{-4wb_{-1}+2wb_0^2+2\sqrt{w}b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\}\right]\times$$

$$\frac{1}{\left[2\left(-3\sqrt{w}b_{-1}b_0 + \sqrt{w}b_0^3 + (-b_{-1}+b_0^2)\sqrt{-4wb_{-1}+wb_0^2} + \{-4\sqrt{w}b_{-1} + 2\sqrt{w}b_0^2 + \right.\right.}$$

$$2b_0\sqrt{-4wb_{-1}+wb_0^2}\}\{\cosh(kx+wt)+\sinh(kx+wt)\} + \{\sqrt{w}b_0 + \sqrt{-4wb_{-1}+wb_0^2}\}$$

$$\left.\left.\{\cosh(2kx+2wt)+\sinh(2kx+2wt)\}\right)\right],$$

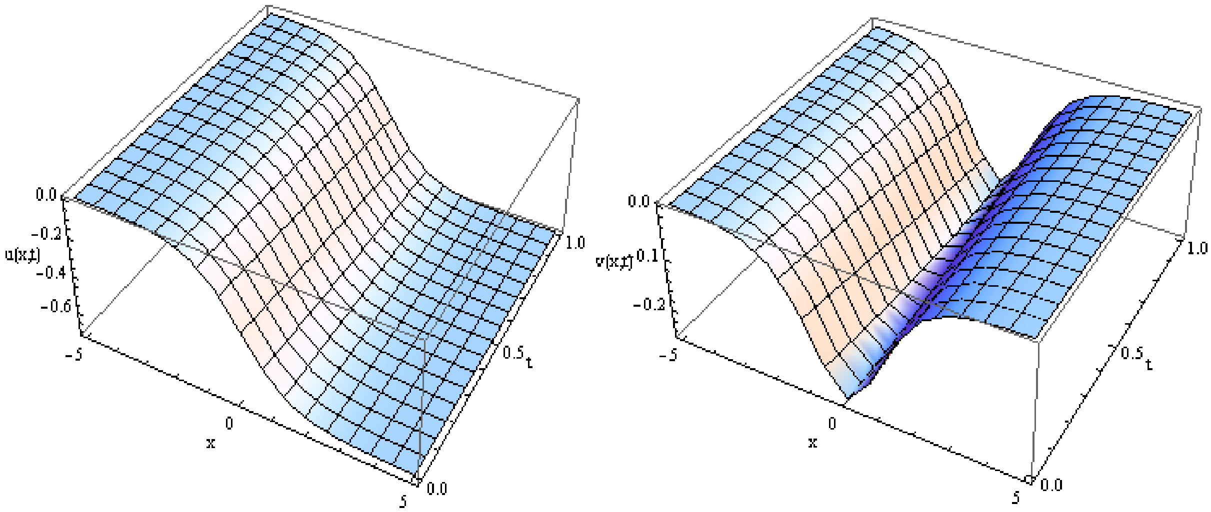


Fig. 3(l). Soliton solutions of eqs. (3.9) and (3.10) in case XII, when $b_0 = 0, b_{-1} = -1, w = 1$.

CONCLUSION

In this chapter, we have successfully implemented Exp-function method developed by He and Wu to derive the exact solutions of the coupled Boussinesq-Burgers equation. The results demonstrate that Exp-function method is straightforward and concise mathematical tool to establish analytical solutions (both solitary and periodic) on nonlinear evolution equations. Therefore, we hope that this method can be more effectively used to investigate other nonlinear evolution equations which are frequently take place in engineering, applied mathematics and physical sciences.

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